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Wave function of a microwave-driven Bose-Einstein magnon condensate

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It has been observed experimentally that a magnon gas in a film of yttrium-iron garnet at room temperature driven by a microwave field exhibits Bose-Einstein condensation (BEC) when the driving power exceeds a critical value. In a previous paper we presented a model for the dynamics of the magnon system in wave-vector space that provides firm theoretical support for the formation of the BEC. Here we show that the wave function of the magnon condensate in configuration space satisfies a Gross-Pitaevskii equation similarly to other BEC systems. The theory is consistent with the previous model in wave-vector space, and its results are in qualitative agreement with recent measurements of the spatial distribution of the magnon condensate driven by a nonuniform microwave field.

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The study of Bose-Einstein condensation (BEC) gained new impetus in 1995 with its discovery in dilute atomic gases.[1] Since then BEC has been observed in solid-state systems at very low temperatures.[2,3] Recently Demokritov *et al.*[4] reported strong experimental evidence of the formation of quasiequilibrium BEC of magnons at room temperature in films of yttrium-iron garnet (YIG) driven by microwave radiation. The initial reports drew criticism by some authors arguing that the process was essentially the same as the dynamical instabilities theoretically described and observed half a century ago, such as the parallel pumping process.[5] Further experiments with resolution in time, frequency and energy[6] clearly demonstrated that the physical phenomena underlying the two processes are very different. In the parallel pumping instability spin waves are driven directly by the microwave field applied parallel to the static field when the microwave power exceeds a threshold value. These waves have opposite wave vectors and frequency which is half the pumping frequency. On the other hand, in the Demokritov's experiments the formation of the BEC magnons occurs in a process with several steps: first primary spin waves are driven parametrically in the film with short microwave pulses with power exceeding the parallel pumping threshold; then the energy of these primary magnons redistributes in about 50 ns through modes with lower frequencies down to the minimum frequency of the dispersion relation for the YIG film as a result of magnon interactions that conserve the number of magnons. This produces a hot magnon gas that remains decoupled from the lattice for several hundred ns due to the long spin-lattice relaxation time. As the microwave power is increased the chemical potential rises and approaches the minimum energy producing an overpopulation of magnons around that frequency. Only if the power exceeds a second threshold there is a condensation of magnons in a narrow region in phase space around the minimum.[6]

In an earlier paper[7] we presented a theoretical model that provides rigorous support for the experimental observation of BEC of magnons in microwave driven YIG films. The theory describes the dynamics of magnons in wave-vector space and leads to the basic features of a BEC; namely, (a) the onset of the BEC is characterized by a phase transition that takes place as the microwave power p is increased and exceeds a critical value p_{c2}; (b) the magnons in the condensate are in coherent states and as such they have nonzero small-signal transverse magnetization that is the order parameter of the BEC; and (c) for $p > p_{c2}$ the magnon system separates into two parts, one in thermal equilibrium with the reservoir and one with N_0 coherent magnons having frequencies and wave vectors in a very narrow region of phase space. As the microwave power increases further N_0 approaches the total number of magnons pumped into the system. Although the theory explains the experimental observations and provides support for the formation of the BEC it gives no information on the spatial distribution of the condensate. In this Rapid Communication we show that the wave function of the magnon condensate in configuration space satisfies a Gross-Pitaevskii equation similarly to other BEC systems. The theory is consistent with the previous model in wave-vector space and its results are in qualitative agreement with very recent measurements of the spatial distribution of the magnon condensate driven by a nonuniform microwave field.[8]

To study the dynamics of the magnon system in wave-vector space we have used[7] the following Hamiltonian:

$$H = H_0 + H^{(4)} + H'_{eff}(t), \tag{1}$$

where

$$H_0 = \hbar \sum_k \omega_k c_k^+ c_k \tag{2}$$

is the unperturbed Hamiltonian for free magnons with frequency ω_k and wave vector $\vec{k}$ described by creation and annihilation operators c_k^+ and c_k. A key feature of the dispersion relation in a film magnetized in the plane is the minimum frequency at a finite k resulting from the combined effects of exchange and magnetic dipolar interactions. In a 5-μm-thick YIG film the minimum frequency ω_{k0} occurs at $k_0 \sim 10^5\ \text{cm}^{-1}$. In Eq. (1) $H^{(4)}$ represents the nonlinear magnetic interactions and $H'_{eff}(t)$ the effective driving of $\vec{k}$, $-\vec{k}$ magnon pairs through the collective action of the secondary magnons resulting from the microwave pumping and bunched in the states with frequency close to ω_{k0}.[7]

In the process studied here the term $H^{(4)}$ in Eq. (1) representing the four-magnon interaction has contributions conserving energy and momentum given by[9,10]

$$H^{(4)} = \hbar \sum_{k_1 k_2 k_3 k_4} \frac{1}{2} T_{1234} c_{k1}^{+} c_{k2}^{+} c_{k3} c_{k4} \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) + \text{H.c.}, \tag{3}$$

where the interaction coefficients are determined mainly by the dipolar and exchange energies. For the k values relevant to the experiments the exchange contribution is negligible so that the coefficients in Eq. (3) are dominated by the dipolar interaction and given approximately by[9] $T_{1234} = \omega_M / NS$, with N being the number of spins S and $\omega_M = \gamma 4\pi M$, where M is the magnetization and $\gamma = g\mu_B/\hbar$ is the gyromagnetic ratio (2.8 GHz/kOe for YIG). In Eq. (1) $H'_{eff}(t)$ is the Hamiltonian for driving $\vec{k}$, $-\vec{k}$ magnon pairs with frequency close to ω_{k0} given by[7]

$$H'_{eff}(t) \cong \hbar (h\rho)_{eff} e^{-i2\omega_{k0}t} c_k^{+} c_{-k}^{+} + \text{H.c.}, \tag{4}$$

where

$$(h\rho)_{eff} = -i\eta_m [(p - p_{c2})/(p_{c2} - p_{c1})]^{1/2} \tag{5}$$

represents an effective driving field related to the microwave pumping power p, the critical power p_{c2} for the formation of the BEC, the critical power p_{c1} for parallel pumping and the intermagnon relaxation rate η_m. Note that at first sight driving Hamiltonian (4) seems not to conserve the number of magnons. However Eq. (4) results from Eqs. (41) and (42) of Ref. 7 expressing four-magnon processes in which two magnons from the reservoir are destroyed and a $\vec{k}$, $-\vec{k}$ magnon pair is created.

Using Hamiltonian (1) with Eqs. (3) and (6) as the interaction and driving terms one can write the Heisenberg equation for the magnon operator c_k as

$$i\frac{dc_k}{dt} = (\omega_k - i\eta_m)c_k - |h\rho|_{eff} e^{-i2\omega_{k0}t} c_{-k}^{+} + V_{(4)} \sum_{k_2 k_3 k_4} c_{k2}^{+} c_{k3} c_{k4} \delta(\vec{k} + \vec{k}_2 - \vec{k}_3 - \vec{k}_4), \tag{6}$$

where $V_{(4)} = 4T_{1234} = 4\omega_M/NS$, the summation is carried out in wave vectors $\vec{k}$, $-\vec{k}$, and the relaxation rate η_m assumed to be the same for the wave vectors involved was introduced phenomenologically. If one assumes that only one pair mode $\vec{k}_0$, $-\vec{k}_0$ is driven by the collective action, Eq. (6) and the corresponding one for the operator c_{-k0}^{+} can be solved in steady state to give the population of the k_0 mode driven by the effective field and saturated by the four-magnon interaction,[7]

$$n_{k_0} = n_H \frac{(p - p_{c2})^{1/2}}{(p_{c2} - p_{c1})^{1/2}}, \tag{7}$$

where $n_H \equiv \eta_m / 2V_{(4)} = \eta_m NS / 8\omega_M$. Equation (7) shows that the k_0 mode is driven collectively only if the microwave power exceeds the threshold value p_{c2}. In the single-mode approximation we assume that neighboring modes in a narrow region of phase space around k_0, ω_{k0} behave similarly to the k_0 mode but with a smaller population and neglect the interaction between modes. They form a condensate with a total number of magnons,

$$N_0 = p_{k0} n_H \frac{(p - p_{c2})^{1/2}}{(p_{c2} - p_{c1})^{1/2}}, \tag{8}$$

where p_{k0} is a factor representing the number of states occupied by condensate magnons weighted relative to the k_0 mode.

In Ref. 7 it is also shown that for $p \geq p_{c2}$ the k_0 magnons pumped up out of equilibrium by the collective action are in a coherent magnon state. Coherent magnon states are defined[11,12] as the eigenkets of the circularly polarized magnetization operator $m^{+} = m_x + i m_y$, where m_x and m_y are the small-signal transverse magnetization operators and z is the equilibrium direction of the magnetization. They can be written as the direct product of single-mode coherent states, the eigenstates of the annihilation operator, $c_k|\alpha_k\rangle = \alpha_k|\alpha_k\rangle$, the eigenvalue α_k being a complex number. Contrary to the eigenstates of the unperturbed Hamiltonian that have zero expectation values for the transverse magnetization and well defined number of magnons, the coherent states do not have well defined number of magnons but they have nonzero expectation values for the magnetization m^{+} with a well defined phase. It can be shown[11–13] that the coherent state $|\alpha_k\rangle$ has an expectation value of the number operator given by $\langle n_k\rangle = |\alpha_k|^2$ and a small-signal transverse magnetization,

$$m^{+} \propto |\alpha_k| = \langle n_k \rangle^{1/2}. \tag{9}$$

The results of the one-mode theory of Ref. 7 agree quite well with the experimental data of Refs. 4 and 6 and provide rigorous support for the formation of the quasiequilibrium BEC of magnons. However it falls short in describing the spatial characteristics of the condensate, which is the aim of this Rapid Communication. In BEC systems the condensate wave function is proportional to the order parameter.[1,14,15] So we define the wave function for the BEC of magnons as

$$\psi(\vec{r},t) = \frac{1}{V^{1/2}} \sum_k e^{i\vec{k}.\vec{r}} \alpha_k. \tag{10}$$

where α_k are the coherent-state eigenvalues for k states around k_0 and V is the volume of the sample. It is important to note that wave function (10) is a function of time and space and since $|\alpha_k| = \langle n_k\rangle^{1/2}$ it implies that

$$\int d^3r \psi^* \psi = \sum_k n_k = N_0, \tag{11}$$

where N_0 is the number of magnons in the condensate. From Eq. (6) one can then obtain an equation of motion for the wave function following the same steps employed in the study spin-wave solitons.[16–18] First we consider that wave function (10) corresponds to a narrow packet in wave-vector space with a central wave number k_0 and central frequency ω_{k0}, so that the frequency of the states involved can be written as

$$\omega_k \cong \omega_{k0} + \lambda_x k_x^2 + \lambda_z k_z^2, \tag{12}$$

where $\lambda_x=[\partial^2\omega_k/\partial k_x^2]_{k0}/2$ and $\lambda_z=[\partial^2\omega_k/\partial k_z^2]_{k0}/2$, with x and z being the coordinates in the film plane. Since the central mode corresponds to a pair $\vec{k}_0$, $-\vec{k}_0$, one can introduce a slowly varying envelope $\psi_0(\vec{r},t)$ in a frame rotating with frequency ω_{k0} such that

$$\psi(\vec{r},t) = 2\cos(\vec{k}_0 \,.\, \vec{r})\psi_0(\vec{r},t)e^{-i\omega_{k0}t} \tag{13}$$

where

$$\psi_0(\vec{r},t) = \frac{e^{i\omega_{k0}t}}{V^{1/2}}\sum_{\delta k} e^{i\vec{\delta k}.\vec{r}}\alpha_k \tag{14}$$

and $\vec{k}=\vec{k}_0+\delta\vec{k}$. Considering that the four-magnon interaction process must conserve energy, one can do calculations similar to those in spin-wave solitons[16,17] using Eqs. (12)–(14) to show that Eq. (6) leads to

$$i\frac{\partial\psi_0}{\partial t} = -i\eta_m\psi_0 - \lambda_x\frac{\partial^2\psi_0}{\partial x^2} - \lambda_z\frac{\partial^2\psi_0}{\partial z^2} + 2V_{(4)}V|\psi_0|^2\psi_0 - |h\rho|_{eff}\psi_0^*. \tag{15}$$

This equation has the form of the celebrated Gross-Pitaevskii (GP) equation used to describe the wave function of other BEC systems.[1,14,15] Without the last term it is also the nonlinear Schrödinger equation used to study solitons.[16–18] Note that the usual GP equation has a term describing the non uniform external potential, such as the trapping potential in atomic gas systems.[1,14,15] This is not present in Eq. (15) which has, instead, a non uniform driving term $|h\rho|_{eff}\psi_0^*$ due to the spatial variation of the pumping microwave field. In the remainder of this Rapid Communication we will assume that the pumping is uniform in the x direction to reduce the problem to a one dimensional case. This is indeed the situation in the experiments of Ref. 8 where a thin wire or tape placed on top and close to the YIG film is used to generate the driving magnetic field. In order to check the consistency of the current approach with the results obtained from the analysis of the dynamics in wave-vector space let us obtain the solution of Eq. (15) in steady state assuming a uniform driving field. With $d/dt=d/dx=d/dz=0$ in Eq. (15) we obtain,

$$(2V_{(4)})^2V|\psi_0|^2|\psi_0|^2 = |h\rho|_{eff}^2 - \eta_m^2. \tag{16}$$

Integrating Eq. (16) in the volume V and using Eqs. (5) and (11), one can see that the number of magnons in the condensate is the same as in Eq. (8) as long as the factor p_{k0} obeys the relation

$$\int d^3r|\psi_0|^2|\psi_0|^2 = \frac{N_0^2}{p_{k0}^2V}. \tag{17}$$

Actually Eq. (17) provides a formal definition of the factor p_{k0} which was introduced in an *ad hoc* manner in Ref. 7. One can now write the wave function in the same form as in other BEC systems[1,14,15]

$$\psi_0 = \sqrt{N_0}\chi(z), \tag{18}$$

where $\chi(z)$ is a normalized wave function independent of the number of magnons in the condensate. To find $\chi(z)$ we consider Eq. (15) without the driving and relaxation terms and assume that their effect is expressed in the number of magnons in the condensate given by Eq. (8). With $d/dt=d/dx=0$ in Eq. (15) we obtain

$$\lambda_z\frac{d^2\chi}{dz^2} - b|\chi|^2\chi = 0, \tag{19}$$

where the coefficient b obtained with Eq. (8) considering $p_{c2} \gg p_{c1}$ becomes

$$b = \eta_m(p/p_{c2} - 1)^{1/2}. \tag{20}$$

Notice that there is a characteristic length associated with Eq. (19) given by

$$\xi = (\lambda_z/b)^{1/2}, \tag{21}$$

which is identified as the healing length.[15] With parameter values appropriate for the YIG film used in the experiments of Refs. 4 and 6, $H=1$ kOe, $4\pi M=1.76$ kG, exchange stiffness $D=2\times10^{-9}$ Oe.cm^2 we obtain from the dispersion relation $\lambda_z=0.13$ s^{-1} cm^2. Assuming $p=2p_{c2}$ and using $\eta_m=5\times10^7$ s^{-1} we find for the healing length $\xi=0.5$ μm. This value is very small compared to the typical length scale of the spatial variation of the driving field. Thus we can use Eqs. (19) and (20) to calculate the spatial distribution of the condensate for a non uniform pumping power $p(z)$. This has been done for a parallel pumping field produced by a microwave current in a thin wire with axis in the x-direction at a distance s from the film. In this case the z component of the field in the film plane is easily obtained using Ampere's law resulting in

$$\frac{p(z)}{p_{\max}} = \left(\frac{h(z)}{h_{\max}}\right)^2 = \frac{s^2}{z^2+s^2}, \tag{22}$$

where $z=0$ is the projection of the wire axis on the plane. The profiles of the driving field and the power given by Eq. (22) are shown in the inset of Fig. 2 for $s=32$ μm, which is the distance inferred from the figures in Ref. 8. We have used numerical integration of Eq. (19) to find the wave function $\chi(z)$ for the spatially varying pumping power given by Eq. (22) assuming that $p_{\max}$ is the applied microwave power P. Figure 1 shows the magnon condensate density, expressed by the wave function squared, calculated for $s=32$ μm, critical power for BEC formation $p_{c2}=0.8$ W inferred from the data of Ref. 8, and two values of the pumping power, $P=1$ and 6 W. The magnon density is peaked at $z=0$ where the power is maximum and falls to zero outside the active region defined by $p(z)>p_{c2}$. The shape of the magnon density agrees qualitatively with the spatial distribution of the condensate measured experimentally with Brillouin light scattering.[8] The agreement is especially good in the central part of the distribution. However while the theoretical density falls off to zero abruptly at $z=w/2$, the value for which $p(w/2)=p_{c2}$, the experimental distribution decays away slowly outside the active region.

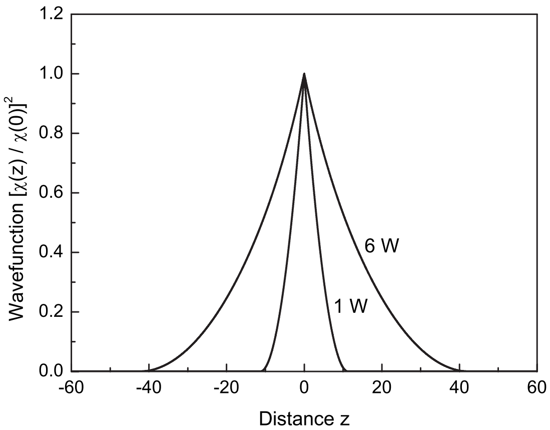

FIG. 1. Condensate magnon wave function calculated with Eq. (19) for microwave driving with a wire with axis at a distance $s = 32\ \mu$m from the YIG film for two values of pumping power $P = 1$ and 6 W.

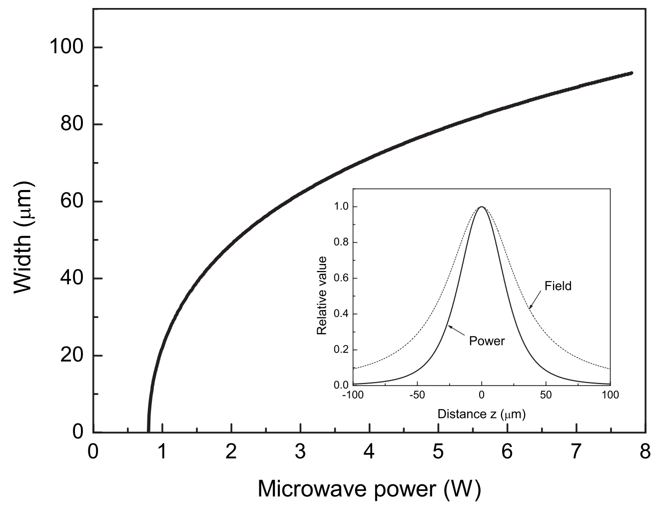

FIG. 2. Width of the active BEC pumping region as a function of microwave power for driving with a wire with axis at a distance $s = 32\ \mu$m from the YIG film. Inset shows the field and power profiles used in the calculation.

Figure 2 shows the width w of the active region calculated with $p(w/2) = p_{c2}$ as a function of the microwave power for the same wire configuration considered before. The increase of the width with pumping power is in agreement with the experimental measurements of the spatial width of the condensate.[8] However the measured widths are larger than the theoretical values by tens of μm. This is attributed to the fact that the parametric and secondary magnons propagate away from the pumping region before and during the process of BEC formation. In fact the group velocity of magnons with frequency 4 GHz propagating perpendicularly to the static field obtained from the dispersion relation is 42 μm/μs, a number that comfortably explains the discrepancy between theory and experiments.

In summary we have developed a theory for the spatial wave function of the Bose-Einstein condensate of magnons in a microwave driven YIG film that is consistent with the model for the dynamics in wave-vector space previously presented.[7] The calculated spatial BEC magnon density is in qualitative agreement with recent measurements[8] of the condensate spatial distribution in the case of microwave driving produced by a thin wire close to the film.

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